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COMPLEMENTARY METHODS IN THE PROBLEMS OF THE STATE OF STRESS IN SHELLS OF REVOLUTION

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Minimum energy and orthogonal projection methods are discussed for the problems of the state of stress in orthotropic shells of revolution, of variable thickness. The methods provide approximate solutions and make it possible to estimate their error in the energetic norm.

The formulation of the orthogonal projections method which is the complementary of the minimum energy method, makes possible the estimation of the errors of the approximate solutions in the energetic norm. It was given in [1], where the above methods were also used in connection with the three-dimensional problems of the theory of elasticity, and for plates of constant thickness.

1. Fundamental relations. The potential energy of deformation of an orthotropic shell can be written in the form [2]

$$W(\omega)=\frac{1}{2}\int_{\Omega}(C_{11}\varepsilon_{11}^2+2C_{12}\varepsilon_{11}\varepsilon_{22}+C_{22}\varepsilon_{22}^2+C_{66}\varepsilon_{12}^2)\,d\Omega+ \qquad (1.1)$$

$$\frac{1}{2}\int_{\Omega}(D_{11}\gamma_{11}^2+2D_{12}\gamma_{11}\gamma_{22}+D_{22}\gamma_{22}^2+4D_{66}\gamma_{12}^2)\,d\Omega$$

Here $\omega=(u, v, w)$ denotes the displacements of a point of the middle surface of the shell, the displacements being functions of φ and z and 2π -periodic in φ; $z \in [0, L]$, L denotes the length of the shell, (r, φ, z) are cylindrical coordinates, ε_{ik} and γ_{ik} are the deformation components of the shell of revolution expressed by ω, by the coefficients A_1^2, A_2^2 of the first quadratic form, the radii of curvature R_1, R_2 and by the generatrix $r(z)$ [3]. Finally, the coefficients C_{ik} and D_{ik} depend on the shell thickness $h(\varphi, z)$, moduli of elasticity E_1 and E_2, Poisson's ratios ν_1 and ν_2 and on the shear modulus G [2].

2. Basic assumptions. When the functions u, v and w are normed, the region $\Omega=(0, 2\pi)\times(0, L)$ is regarded as the domain of definition of these functions.

We assume that the functional $W(\omega)$ defined by (1.1) is specified on some subspace (defined below) of the space $H_0=W_{2,0}^1(\Omega)\times W_{2,0}^1(\Omega)\times W_{2,0}^2(\Omega)$ which is a straight product of the Sobolev [4] spaces of φ-periodic functions; $\omega=(u, v, w) \in H_0$, $u \in W_{2,0}^1(\Omega)$, $v \in W_{2,0}^1(\Omega)$, $w \in W_{2,0}^2(\Omega)$. The displacement function ω satisfies certain boundary conditions which can be written in the form

$I\omega = 0$ where I is a boundary condition operator acting in a space of functions defined on S_1

$$S_1 = S_{11} \cup S_{12}$$
$$S_{11} = \{(\varphi, z) \mid 0 < \varphi < 2\pi, \quad z = 0\}$$
$$S_{12} = \{(\varphi, z) \mid 0 < \varphi < 2\pi, \quad z = L\}$$

We assume that the following conditions hold:

1) $h(\varphi, z)$ is a function measurable on $d\Omega$ and satisfying almost everywhere in Ω the condition $0 < h_1 \leqslant h(\varphi, z) \leqslant h_2$ where h_1 and h_2 are positive constants;

2) ν_1 and ν_2 are constants, and $0 < \nu_1 < 1$, $0 < \nu_2 < 1$

3) E_1, E_2 and G are positive constants;

4) Function $r(z)$ is twice continuously differentiable on the interval $[0, L]$ and satisfies the following inequalities when $\forall z \in [0, L]$:

$$r(z) \geqslant c, \quad c = \text{const} > 0$$
$$|R_1^{-1} - R_2^{-1}| = |A_1^{-3} \frac{d^2r}{dz^2} + r^{-1}A_1^{-1}| \geqslant c_1, \quad c_1 = \text{const} > 0$$

A generalized derivative $d^3r / dz^3 \in L_\infty(0, L)$ exists;

5) The boundary conditions operator I is a linear continuous mapping from H_0 onto $(L_2(S_1)^m)$ $(m = 3, 4)$ and for $\forall \omega \in H_0$ the condition $W(\omega) = 0$, $I\omega = 0$ implies that $\omega = 0$. In particular, condition 5) will hold if the operator I corresponds to clamping of the shell in the sense that the latter cannot experience any rigid displacements.

We denote by H the closure on the norm

$$\|\omega\|_H^2 = \|u\|^2_{W_2^1(\Omega)} + \|v\|^2_{W_2^1(\Omega)} + \|w\|^2_{W_2^2(\Omega)} \tag{2.1}$$

of the set of φ-periodic functions differentiable in the strip $0 \leqslant z < L$, $-\infty < \varphi < \infty$ and satisfying the condition $I\omega = 0$. Obviously $H \subset H_0$.

Let us consider, in H, the following symmetrical bilinear form:

$$a(\omega', \omega'') = \frac{E_1}{1 - \nu_1\nu_2} \int_\Omega \Big\{ h \Big[\varepsilon_{11}'\varepsilon_{11}'' + \nu_2(\varepsilon_{11}'\varepsilon_{22}'' + \varepsilon_{22}'\varepsilon_{11}'') + \frac{\nu_2}{\nu_1}\varepsilon_{22}'\varepsilon_{22}'' + \frac{1 - \nu_1\nu_2}{E_1} G\varepsilon_{12}'\varepsilon_{12}'' \Big] + \frac{h^3}{12}\Big[\gamma_{11}'\gamma_{11}'' + \nu_2(\gamma_{11}'\gamma_{22}'' + \gamma_{22}'\gamma_{11}'') + \frac{\nu_2}{\nu_1}\gamma_{22}'\gamma_{22}'' + 4\frac{1 - \nu_1\nu_2}{E_1} G\gamma_{12}'\gamma_{12}''\Big]\Big\} d\Omega \tag{2.2}$$

Here ε_{ik}', γ_{ik}' and ε_{ik}'', γ_{ik}'' denote the components of the deformations generated by the displacements ω' and ω''.

By virtue of the assumptions 1) –4), the form a $a(\omega', \omega'')$ is defined for any ω', ω'' on H. It is also clear that $a(\omega, \omega) = 2W(\omega)$. Using the assumption 5) we find, that the conditions $\omega \in H$, $a(\omega, \omega) = 0$ imply that $\omega = 0$. Therefore the form $a(\omega', \omega'')$ generates a scalar product and norm in H, defined by the expression

$$\|\omega\|_{H'}^2 = a(\omega, \omega) \tag{2.3}$$

The following assertion can be proved in an analogous manner [3]:

T h e o r e m *1.* Let the assumptions 1) —5) hold. Then the norms defined by the expressions (2.1),(2.3) are equivalent in the space H, i.e. constants m_1, $m_2 > 0$, exist such that

$$m_1 \|\omega\|_H \leqslant \|\omega\|_{H'} \leqslant m_2 \|\omega\|_H \quad \forall \omega \in H \tag{2.4}$$

Theorem 1 establishes the coercivity of the operator of the theory of shells. Other results connected with coercivity of the operators of shells are given in [5].

3. P r o b l e m o f t h e s t a t e o f s t r e s s i n a s h e l l. Let $g(\varphi, z)$ be the vector of external load acting on the shell. We assume that $g \in H^*$ where H^* is a space conjugate to H. We denote the general solution of the problem of the state of stress of a shell of revolution by the function $\omega_0 \in H$ for which the condition

$$a(\omega_0, h) = (g, h) \quad \forall h \in H \tag{3.1}$$

holds. We know [1] that a solution of the problem (3.1) exists and imparts a minimum to the functional

$$\psi(\omega) = a(\omega, \omega) - 2(g, \omega), \quad \omega \in H$$

If V_k is a finite-dimensional subspace in H, then a unique function $\omega_k \in V_k$ exists for which [1]

$$\psi(\omega_k) = \inf_{\omega \in V_k} \psi(\omega) \tag{3.2}$$

and the following relations hold:

$$\|\omega_k - \omega_0\|_{H'}^2 = \|\omega_0\|_{H'}^2 - \|\omega_k\|_{H'}^2 \tag{3.3}$$

$$\psi(\omega_0) = -\|\omega_0\|_{H'}^2, \quad \psi(\omega_k) = -\|\omega_k\|_{H'}^2 \tag{3.4}$$

It is clear from (3.3) that if the quantity $\|\omega_0\|_{H'}{}^2$ or at least its upper bound, is known, then the error of the approximate solution ω_k can be estimated. To find the upper bound of $\|\omega_0\|_{H'}{}^2$, we use the method of orthogonal projections.

4. M e t h o d o f o r t h o g o n a l p r o j e c t i o n s. Let us denote by E the straight product of six spaces $L_2(\Omega)$, i.e. $E = (L_2(\Omega))^6$. E is a set of all

possible ordered elements of the type

$$(T_1, T_2, T_{12}, M_1, M_2, M_{12}) = M; \quad T_1, T_2, T_{12}, M_1, M_2, M_{12} \in L_2(\Omega)$$

The set E becomes a Hilbert space after introducing in it a scalar product and the norm, assuming that

$$(M', M'') = \int_\Omega (T_1'T_1'' + T_2'T_2'' + T_{12}'T_{12}'' + M_1'M_1'' + M_2'M_2'' + M_{12}'M_{12}'')\, d\Omega \tag{4.1}$$

$$\| M \| = (M, M)^{1/2}$$

If the assumptions 1) —5) hold, we introduce, in the space E, the bilinear symmetric form

$$b(M', M'') = \int_\Omega \frac{1}{h}\Big[D\Big(X_1'X_1'' + \nu_1 X_1'X_2' + \nu_1 X_1''X_2'' + \frac{\nu_2}{\nu_1} X_2'X_2''\Big) + \frac{T_{12}'T_{12}''}{G}\Big] d\Omega + \int_\Omega \frac{12}{h^3}\Big[D\Big(Y_1'Y_1'' + \nu_1 Y_1'Y_2' + \nu_1 Y_1''Y_2'' + \frac{\nu_2}{\nu_1} Y_2'Y_2''\Big) + 4\,\frac{M_{12}'M_{12}''}{G}\Big] d\Omega \tag{4.2}$$

$$X_1 = T_1 - T_2\nu_1, \quad X_2 = T_2 - T_1\nu_2$$

$$Y_1 = M_1 - M_2\nu_1, \quad Y_2 = M_2 - M_1\nu_2, \quad D = E_1^{-1}(1 - \nu_1\nu_2)^{-1}$$

T h e o r e m 2. When the assumptions 1) —3) hold, the form $b(M', M'')$ defines the scalar product in E and the norm

$$\| M \|_E = [b(M, M)]^{1/2} \tag{4.3}$$

equivalent to the basic norm (4.1) of the space E.

P r o o f. Using the inequality $a^2 - b^2 \leqslant -2ab$ and the assumptions 1) —3), we obtain

$$b(M, M) = \int_\Omega \frac{1}{h}\Big[P(T_1, T_2) + \frac{T_{12}^2}{G}\Big] d\Omega + \int_\Omega \frac{12}{h^3}\Big[P(M_1, M_2) + \frac{4M_{12}^2}{G}\Big] d\Omega \geqslant \int_\Omega \frac{1}{h}\Big[Q(T_1, T_2) + \frac{T_{12}^2}{G}\Big] d\Omega + \int_\Omega \frac{12}{h^3}\Big[Q(M_1, M_2) + \frac{4M_{12}^2}{G}\Big] d\Omega \geqslant$$

$$C_1 \int_\Omega (T_1^2 + T_2^2 + T_{12}^2 + M_1^2 + M_2^2 + M_{12}^2)\, d\Omega = C_1 \| M \|^2,$$

$$\forall M \in E$$

$$P(x, y) = \frac{1}{E_1}\Big(x^2 + \frac{\nu_2}{\nu_1} y^2 - 2\nu_1 xy\Big), \quad Q(x, y) =$$

$$\frac{1}{E_1}\left[(1-\nu_1)x^2+\nu_1\left(\frac{1}{\nu_2}-1\right)y^2\right], \quad C_1=\mathrm{const}>0$$

From the assumptions 1) —3) follows the inverse estimate

$$b\,(M,M)\leqslant c_2\,\|M\|^2, \quad \forall M\in E; \quad c_2=\mathrm{const}>0$$

and this proves the theorem.

Let us introduce a linear bounded operator U acting from E into H^*, defined by the expression

$$(UM,\omega)=\int\limits_{\Omega}(T_1\varepsilon_{11}+T_2\varepsilon_{22}+T_{12}\varepsilon_{12}+M_1\gamma_{11}+ M_2\gamma_{22}+2M_{12}\gamma_{12})\,d\Omega, \quad M\in E, \quad \omega\in H \tag{4.4}$$

and denote by F_2 the kernel of the operator U

$$F_2=\{M\,|\,M\in E, \quad UM=0\} \tag{4.5}$$

Here F_2 is a closed linear set in E, i.e. a subspace in E. The linearity of F_2 is obvious, and its closure follows from the fact that F_2 is a submapping of a closed set consisting of a single point (null element in H^*) when the mapping of U is continuous. Consider the operator

$$A:\omega\to A\omega=\{C_{11}\varepsilon_{11}+C_{12}\varepsilon_{22},\quad C_{12}\varepsilon_{11}+C_{22}\varepsilon_{22},\quad C_{66}\varepsilon_{12}, \quad D_{11}\gamma_{11}+D_{12}\gamma_{22},\quad D_{12}\gamma_{11}+D_{22}\gamma_{22},\quad 2D_{66}\gamma_{12}\} \tag{4.6}$$

which represents a linear continuous mapping of H onto E. Let us denote by F_1 the image of the operator A, $F_1=A\,(H)$. Clearly, F_1 is a linear set. We shall show that F_1 is a closed set in E. Let $\{A\omega_n\}$ denote the basic sequence of elements of F_1. By virtue of the completeness of the space E, there exists an element $M^{(0)}\in E$ such that

$$\lim_{n\to\infty}\|A\omega_n-M^{(0)}\|_E=0 \tag{4.7}$$

It remains to confirm that $M^{(0)}\in F_1$. The relations (2.2),(2.3),(4.2),(4.3) and (4.6) imply the following identity:

$$\|A\omega\|_E^2=\|\omega\|_{H'}^2 \quad \forall\omega\in H \tag{4.8}$$

From (4.8) it follows that $\{\omega_n\}$ is the basic sequence in H which converges, by virtue of the completeness of H, to the element $\omega_0\in H$. Now, taking into account (4.7) and remembering that A is a continuous operator from H into E, we obtain

$$\lim_{n\to\infty}A\omega_n=A\omega_0=M^{(0)}$$

Consequently $M^{(0)} \in F_1$ and F_1 is a subspace of E.

Let M denote any element of F_2 and $N = A\omega$ be any element of F_1. From (4.2), (4.5) and (4.6) it follows that

$$b(M, N) = (UM, \omega) = 0$$

therefore F_1 and F_2 are orthogonal subspaces.

Further, we shall show that the subspaces F_1 and F_2 form an expansion E, i.e. $E = F_1 \oplus F_2$. Let us return to the form $a(\omega', \omega'')$ defined by the relation (2.2). The form can be written as follows:

$$a(\omega', \omega'') = (B\omega', \omega''), \quad \forall \omega', \omega'' \in H \tag{4.9}$$

where B is a continuous linear mapping from H into H^* and $(B\omega', \omega'')$ is a scalar product of the elements $B\omega' \in H^*$, $\omega'' \in H$. From the relations (2.2), (4.4) and (4.6) it follows that B is a composition of the mappings $A \in L(H, E)$ and $U \in L(E, H^*)$

$$B = U \circ A \tag{4.10}$$

Let M be any element of E. We shall show that it can be represented in the form $M = M^{(1)} + M^{(2)}$ where $M^{(1)} \in F_1$ and $M^{(2)} \in F_2$. Consider the problem of finding a function $\omega \in H$ such that

$$B\omega = UM \tag{4.11}$$

From (4.9) it follows that the problem (4.11) has a unique solution $\omega \in H$. Then $M^{(1)} = A\omega \in F_1$ and by virtue of (4.10) and (4.11) the following relation holds for the element $M^{(2)} = M - M^{(1)} = M - A\omega$:

$$UM^{(2)} = U(M - A\omega) = 0$$

and from this it follows that $M^{(2)} \in F_2$ and $E = F_1 \oplus F_2$.

Let us now return to the problem of the state of stress of a shell (3.1). From (2.2), (4.4) and (4.6) it follows that the problem (3.1) can be written in the form

$$U \circ A\omega = B\omega = g, \quad g \in H^* \tag{4.12}$$

Let M' be an element of E such that $UM' = g$. Consider the problem of minimizing the functional

$$J(M) = \| M' - M \|_E^2, \quad M \in F_2 \tag{4.13}$$

Since F_2 is a closed linear set in a Hilbert space E, there exists a unique moment $M^{(0)} \in F_2$ for which

$$J(M^{(0)}) = \inf_{M \in F_2} J(M) \tag{4.14}$$

and $M^{(0)}$ is a projection of the element M' onto F_2. Then $M' - M^{(0)} \in F_1$, and a function $\omega \in H$ exists such that

$$A\omega = M' - M^{(0)} \tag{4.15}$$

From this we have

$$B\omega = (U \circ A)\,\omega = U\,(M' - M^{(0)}) = g$$

Consequently, if the element $M^{(0)} \in F_2$ minimizes the functional (4.13), then the function $\omega \in H$ for which $A\omega = M' - M^{(0)}$ is a solution of the problem (4.12), i.e. is a generalized solution of the problem of the state of stress of a shell of revolution (in the notation of Sect. 3 $\omega = \omega_0$). Moreover, taking into account (4.8) and (4.15), we have

$$J\,(M^{(0)}) = \| M' - M^{(0)} \|_E{}^2 = \| A\omega \|_E{}^2 = \| \omega \|_{H'}{}^2$$

We express all the results obtained above in the form of the following theorems.

Theorem 3. Let the assumptions 1) – 5) hold, $g \in H^*$, and M' be an arbitrary element of E satisfying the relation $UM' = g$. Then there exists a unique element $M^{(0)}$ satisfying the conditions (4.14), and if $\omega \in H$ is a solution of the problem (4.12), then $A\omega = M' - M^{(0)}$ and

$$\| M' - M^{(0)} \|_E{}^2 = \| \omega \|_{H'}{}^2 \tag{4.16}$$

Theorem 4. Let $\{F_2{}^{(n)}\}_{n=1}{}^{\infty}$ be a sequence of n-dimensional subspaces of the space F_2. A unique element $M^{(n)} \in F_2{}^{(n)}$ exists such that

$$\| M' - M^{(n)} \|_E{}^2 = \inf_{M \subset F_2^{(n)}} \| M' - M \|_E{}^2 \tag{4.17}$$

and if

$$\lim_{n\to\infty} \{ \inf_{N \in F_2^{(n)}} \| N - M \|_E \} = 0, \quad \forall M \in F_2 \tag{4.18}$$

then

$$\lim_{n\to\infty} \| M' - M^{(n)} - A\omega \|_E = 0 \tag{4.19}$$

$$\| M' - M^{(n)} - A\omega \|_E{}^2 \leqslant \| M' - M^{(n)} \|_E{}^2 - \| \omega_k \|_{H'}{}^2 \tag{4.20}$$

$$\| \omega_k - \omega \|_{H'}{}^2 \leqslant \| M' - M^{(n)} \|_E{}^2 - \| \omega_k \|_{H'}{}^2 \tag{4.21}$$

Here ω is a solution of the problem (4.12) and ω_k is an element of V_k satisfying the relation (3.2).

Proof. Taking into account (4.15), we have

$$\| M' - M^{(n)} \|_E{}^2 \geqslant \| M' - M^{(0)} \|_E{}^2 = \| \omega \|_{H'}{}^2$$

This and (3.3) together with the fact that $\omega_0 = \omega$, yields the inequality (4.21). On the other hand, taking into account (4.5) we find that

$$\| M' - M^{(0)} - A\omega \|_E^2 = \| M^{(0)} - M^{(n)} \|_E^2 \tag{4.22}$$

Considering that $M' - M^{(0)} \in F_1$, $M^{(0)} - M^{(n)} \in F_2$ and $F_1 \perp F_2$, we obtain

$$\| M' - M^{(n)} \|_E^2 = \| (M' - M^{(0)}) + (M^{(0)} - M^{(n)}) \|_E^2 = \| M' - M^{(0)} \|_E^2 + \| M^{(0)} - M^{(n)} \|_E^2 \tag{4.23}$$

From (4.16), (4.22) and (4.23) follows

$$\| M' - M^{(n)} - A\omega \|_E^2 = \| M^{(0)} - M^{(n)} \|_E^2 = \| M' - M^{(n)} \|_E^2 - \| M' - M^{(0)} \|_E^2 = \|M' - M^{(n)} \|_E^2 - \| \omega \|_{H'}^2$$

and this, together with (3.4), yields (4.20).

The following relation holds for $\forall M \in F_2^{(n)}$:

$$\| M' - M \|_E^2 = \| M' - M^{(0)} \|_E^2 + \| M^{(0)} - M \|_E^2$$

and this, together with (4.17) and (4.18), yields $\lim_{n\to\infty} \| M^{(0)} - M^{(n)} \|_E^2 = 0$. Now (4.19) follows from (4.22) and this completes the proof of the theorem.

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